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AN INEQUALITY IN THE THEORY OF A SEMILINEAR ELASTIC BODY*

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An inequality for a geometrically non-linear problems is obtained as an analogue of the Prager-Synge identity in linear elasticity theory on the basis of a representation of the elastic energy density of a semilinear elastic body.

The convexity of the potential energy functional in geometrically linear problems of elasticity theory enabled a dual variational problem, the Castigliano principle, to be formulated. The fact that the lower bound of the direct functional I is associated with the lower bound of the dual by the relationship

$$\inf I = -\inf J = \sup(-J) \tag{0.1}$$

turns out to be remarkable here.

The potential energy functional I is examined in a set of kinematically allowable displacement fields $\mathbf{w}$, the dual J in a set of statically allowable stress fields σ. The property (0.1) of the dual problem enables the minimum value of the direct functional $I(\mathbf{w})$ to be estimated as accurately as desired from below. But this would at once yield /1/ an estimate of the approximation $\mathbf{w}$ minimizing the element $\mathbf{w}^\circ$ in the norm L_2

$$\|\mathbf{w}-\mathbf{w}^\circ\|_{L_2(V_0)} \leqslant C\,(I(\mathbf{w})-d)^{1/2} \tag{0.2}$$

where $d \leqslant I(\mathbf{w}^\circ)$ is the lower limit of the minimum value of the functional I, C is a constant, and V_0 is the domain occupied by the elastic body in the undeformed state.

The estimate (0.2) can be reduced to the form /2/

$$\|\bar{\sigma}-\sigma^\circ\|_{L_2(V_0)} \leqslant C\,\|\bar{\sigma}-\sigma'\|_{L_2(V_0)} \tag{0.3}$$

Here $\bar{\sigma}$ is a statically allowable stress field, σ' is the kinematically allowable stress, and σ° is the true state of stress of the elastic body.

The natural desire to extend these results to the case of geometrically non-linear problems of elasticity theory encounters a number of difficulties in principle. The first is associated with the fact that the potential energy functional in geometrically non-linear problems is not convex. In substance, this excludes the possiblity of constructing a dual functional for which condition (0.1) would be satisfied. It is thereby impossible to compute the lower bound of the potential energy functional as exactly as desired. The second difficulty is that the relationship (0.2) is not valid in geometrically non-linear problems. And even in the case when the dual problem /4/ is constructed formally according to standard procedure /3/ and a lower bound of the minimum value of the direct functional is obtained, the connection between this estimate and the error of the approximate solution is not clear.

An attempt is made below to obtain an inequality of the type (0.3) for a semilinear

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elastic body. To this end, a representation is given of the elastic energy density that is identical with the standard one in the domain of small deformations /5/.

1. The elastic energy of a semilinear material. Let ξ^a be the Lagrange coordinates of points of an elastic body that occupies a domain V_0 in the undeformed state and V in the deformed state, $y^i(\xi^a)$ the Cartesian coordinates of points of the elastic body in the undeformed state and $x^i(\xi^a)$ in the deformed state. Furthermore, the Latin subscripts $a, b, c, \ldots$ take the values 1, 2, 3, ... and correspond to projections on the ξ^a coordinate axes in the undeformed state. The complete system of equations of the statics of the theory of a semilinear material can be represented in the form /5/

$$p_{i|a}^{a} = 0, \quad p_i^a n_a|_{\partial V_0} = P_i; \quad p_i^a = \partial U/\partial x_a^i, \quad x_a^i = \partial x^i/\partial \xi^a \tag{1.1}$$

The line in the subscripts denotes the operation of covariant differentiation with respect to the connectedness of the ξ^a coordinate system in the undeformed state. For simplicitiy in the subsequent discussion it is assumed that the elastic body is not clamped and subjected to the action of a "dead" mass $F_i \equiv 0$ and surface P_i forces, and n_a are components of the external normal vector to the boundary of the elastic body V_0. The elastic energy density U for an isotropic semilinear material is given by the formula

$$U = {}^1/_2\lambda\,(g^{\circ ab}\gamma_{ab})^2 + \mu\gamma^{ab}\gamma_{ab}, \quad \gamma_{ab} = |x|_{ab} - \overset{\circ}{g}_{ab} \tag{1.2}$$

where λ and μ are Lamé elastic constants, $g^{\circ ab}$ are contravariant components of the metric tensor of the Lagrange coordinate system in the undeformed state, $|x|_{ab}$ is the distortion modulus $x_a^i : x_a^i = |x|_a{}^b\lambda_b{}^i$, and $\lambda_b{}^i$ satisfies the relations

$$\delta_{ij}\lambda_a{}^i\lambda_b{}^i = \overset{\circ}{g}_{ab}, \quad g^{\circ ab}\lambda_a{}^i\lambda_b{}^j = \delta^{ij}, \quad \det\|\lambda_a{}^i\| = +1 \tag{1.3}$$

Unless otherwise specified, juggling of the indices $a, b, c \ldots$ is performed everywhere later by using the metric $g_{ab}{}^\circ$.

The object p_i^a that is the covector with respect to the transformation of the x^i Cartesian coordinate system and the vector with respect to transformation of the Lagrange coordinates ξ^a in the undeformed state, is called the Piola-Kirchhoff tensor in the geometrically non-linear theory of elasticity.

Let us represent U as a function of $|x|_{ab}$

$$U(x_a{}^i) = \lambda/2\,(g^{\circ ab}|x|_{ab} - 3)^2 + \mu\,(|x|^{ab}|x|_{ab} - 2g^{\circ ab}|x|_{ab} + 3) = V(|x|_{ab}) \tag{1.4}$$

Lemma 1. If $(g^{\circ ab}|x|_{ab} = 3) \leqslant (1-2\nu)/\nu$, ν is Poisson's ratio, then

$$U(x_a{}^i) = \inf_{\varkappa_a{}^i \in (1.3)} \Phi(\varkappa_a{}^i, x_b{}^j), \quad \Phi(\varkappa_a{}^i, x_b{}^j) = V(\varkappa_b{}^i x_{ib})$$

Here and henceforth, writing $\varkappa_a{}^i \in (1.3)$ means that the object $\varkappa_a{}^i$ satifies the constraints (1.3).

To give a foundation to the assertion made, the following lemma proved in /4/ is required.

Lemma 2. Let q^{ab} be an arbitrary tensor and μ^{ab} the components of an orthogonal matrix satisfying the condition $\det\|\mu_b{}^a\| = 1$. Then

$$\sup_{\mu_{ab}} q^{ab}\mu_{ab} = \max_{s_{ab}} \{|q|^{ab}s_{ab}\}$$

where s_{ab} is one of the matrices

$$E, A, B, C \tag{1.5}$$

when $\det\|q^{ab}\| \geqslant 0$ and

$$-E, -A, -B, -C \tag{1.6}$$

when $\det\|q^{ab}\| < 0$. Here $|q|_b{}^a$ is the modulus of the tensor q^{ab}, E is the unit 3x3 matrix, $A = \operatorname{diag}\{1, -1, -1\}$, $B = \operatorname{diag}\{-1, +1, -1\}$, $C = \operatorname{diag}\{-1, -1, 1\}$.

Proof of Lemma 1. We have

$$\Phi = {}^1/_2\lambda\,(g^{\circ ab}\varkappa_a{}^i x_{ib} - 3)^2 + \mu\,(|x|^{ab}|x|_{ab} - 2g^{\circ ab}\varkappa_a{}^i x_{ib} + 3) \tag{1.7}$$

where the identity $g^{\circ ac}g^{\circ ld}\chi_a{}^i x_{lb}\chi_c{}^j x_{jd} = |x|^{ab}|x|_{ab}$ is used.

Hence it follows that

$$\inf_{\varkappa_a{}^i \in (1.3)} \Phi = \inf_y\,[{}^1/_2\lambda\,(y-3)^2 + \mu\,(|x|^{ab}|x|_{ab} - 2y + 3)]; \quad y = g^{\circ ab}\varkappa_a{}^i x_{ib}$$

Therefore, investigation of the minimum value of the function reduces to investigating the minimum of the quadratric trinomial $f(y)$, where $\inf f(y)$ is reached at the point $y_0 = 3 + (1-2\nu)/\nu$. Since $y \leqslant |x|_a{}^a$, in the case when $|x|_a{}^a < y_0$ the minimum Φ in the set of orthogonal matrices $\varkappa_a{}^i$ under consideration is reached at the point $y = |x|_a{}^a$. In the case when $|x|_a{}^a > y_0$ we have

$$\inf_{\varkappa_a{}^i \in (1.3)} \Phi = \inf_{y \in R^1} f(y)$$

Calculations yield

$$\bar{U} = \inf_{\varkappa_a{}^i \in (1.3)} \Phi = \begin{cases} V(|x|_{ab}), & |x|_a{}^a \leqslant y_0 \\ \mu(|x|^{ab}|x|_{ab} - y_0), & |x|_a{}^a > y_0 \end{cases} \tag{1.8}$$

which is what is required.

We see that if Poisson's ratio is $\nu < 1/(2+\gamma)$, where $\gamma = (\gamma^{ab}\gamma_{ab})^{1/2}$, then $U = \bar{U}$. For small strains ($\gamma \sim 10^{-2}$) this is valid for a very broad class of materials. Almost incompressible quantities (the quantity ν close to $^1/_2$) are the exception. For this reason, unless otherwise specified later, when speaking of a similinear material we will understand that its elastic energy density is given by the relationship (1.8).

Let us note that the function U is identical $\gamma_a{}^a > (1-2\nu)/\nu$ apart from a constant with the Treloar elastic potential obtained from statistical considerations regarding the structure of rubber /6/.

2. ***The function Φ and certain properties associated with it.*** Since $\Phi(\varkappa_a{}^i, x_b{}^j) = V(\varkappa_a{}^i x_{ib})$, while the function $V(d_{ab})$ is convex in d_{ab}, the function Φ itself is also convex in $x_a{}^i$ for fixed $\varkappa_a{}^i$.

Consider the functional

$$I(\varkappa_a{}^i, x^j) = \int_{V_0} \Phi \, d\tau - \int_{\partial V_0} P_i x^i d\sigma \tag{2.1}$$

where $d\tau$ is a volume element of the domain V_0 occupied by the elastic body and $d\sigma$ is a surface element of its boundary ∂V_0.

Because of the convexity of Φ in x^i, the problem of stationary points of the functional (2.1) for fixed $\varkappa_a{}^i$ in the set of functions $x^i(\xi^b)$ is identical with the problem regarding its minimum. The variational principle known in the theory of a semilinear material concerning the stationarity of the potential energy in the terminology under consideration is formulated as a problem concerning stationary points in $x^i(\xi^a)$ for the functional

$$I^\circ(x^i) = \inf_{\varkappa_a{}^i \in (1.3)} I(\varkappa_a{}^i, x^j) \tag{2.2}$$

Note that despite the fact that the functional $I(\varkappa_a{}^i, x^j)$ is convex in x^i, the functional (2.2) is not convex in x^i. The circumstance mentioned is related to the fact that a set of orthogonal tensors will not be convex in a linear space of tensors of the second rank that is introduced in a standard manner.

Besides problems of the minimum of the functional I in x^i we consider its dual problem. According to the standard procedure /3/, it is formulated as the problem of the minimum of the functional

$$J(\varkappa_a{}^i, p_j{}^b) = \int_{V_0} \Phi^* d\tau \tag{2.3}$$

in a set of dual variables $p_j{}^b$ satisfying the equations of statics (the first two relations in (1.1)). The function Φ^* is the Young-Fenchel transformation of the function Φ in the distortion $x_a{}^i$. Since the function Φ is convex in x^i, we have

$$\inf_{x^i} I(\varkappa_a{}^i, x^j) = \sup_{p_j{}^b} (-J(\varkappa_a{}^i, p_j{}^b)) \tag{2.4}$$

The Young-Fenchel transformation of the function Φ in $x_a{}^i$ can be evaluated simply because the function is the sum of a quadratric and a linear function. Omitting details, we write down the answer at once

$$\Phi(\varkappa_a{}^i, p_j{}^b) = V^*(\varkappa_a{}^i p_i{}^b) \tag{2.5}$$

where V^* is the Young-Fenchel transformation of the function V in $|x|_{ab}$

$$V^*(r_{ab}) = \frac{1}{4\mu}\left[r^{ab}r_{ab} - \frac{\lambda(\lambda+2\mu)}{(3\lambda+2\mu)^2}(r_a{}^a)^2\right] + r_a{}^a \tag{2.6}$$

Let us prove the fundamental property of the auxiliary variational problem (2.1).

Lemma 3. Let x^{oi} be a stationary point of the functional (2.2) and $\varkappa_b{}^{oi}$ the orthogonal part of its corresponding distortion $x_a{}^{oi} = \varkappa_c{}^{oi}|x|_a{}^c$. Then x^{oi} supplies the minimum of the functional $I(\varkappa_a{}^{oi}, x^i)$ and

$$I(\varkappa_a{}^{oi}, x^{oj}) = I^\circ(x^{oj}) \tag{2.7}$$

which is what required.

Proof. The identity (2.7) is obvious and can be verified in an elementary way.

We will show that $x^{oi}(\xi^b)$ is a stationary point of the functional $I(\varkappa_b{}^{oi}, x^j)$ and therefore, supplies the minimum to it. Indeed, the Euler equations of the functional under consideration have the form

$$p_{i|a}^a = 0, \quad p_i{}^a n_a|_{\partial V_\circ} = P_i, \quad p_i{}^a = \partial\Phi/\partial x_a{}^i \tag{2.8}$$

Since x^{oi} is a stationary point of the functional $I^\circ(x^j)$, it satisfies the first two relationships of (2.8) where $p_i{}^a = \partial U/\partial x_a{}^i|_{x^{\circ i}}$. To prove the assertion it remains to show that

$$\partial U/\partial x_a{}^i|_{x^{\circ i}} = \partial\Phi/\partial x_a{}^i|_{x^{\circ i}} \tag{2.9}$$

Transformation of the right-hand side of (2.9) indeed yields the required result.

Lemma 4. Let $x_0{}^i$ be the minimizing element of the functional $I^\circ(x^i)$ and $\varkappa_{0a}{}^i$ the orthogonal part of its distortion. Then $x_0{}^i$ minimizes the functional $I(\varkappa_{0a}{}^i, x^j)$ and coincides with one of the stationary points of the functional $I^\circ(x^i)$.

The first assertion is almost obvious. Indeed $I^\circ(x_0{}^i) = I(\varkappa_{0a}{}^i, x_0{}^j) \leqslant I(\varkappa_a{}^i, x^j)$ for any $\varkappa_a{}^i, x^j$ by virtue of (2.2). Therefore $I(\varkappa_{0a}{}^i, x_0{}^j) \leqslant I(\varkappa_{0a}{}^i, x^j)$ for any x^j, which proves what is required. Hence it follows that $x_0{}^i$ is a stationary point of the functional $I(\varkappa_{0a}{}^i, x^j)$ and, thereby satisfies (2.6). By virtue of (2.7), the minimizing element $x_0{}^j$ satisfies the system of Eqs.(1.1). And this indeed means that $x_0{}^i$ coincides with one of the stationary points of the functional $I^\circ(x^j)$.

The identity (2.4) is valid for any $\varkappa_a{}^i$. Hence, the relationship

$$\inf_{\varkappa_a{}^i}\inf_{x^i} I(\varkappa_a{}^i, x^j) = \inf_{\varkappa_a{}^i}\inf_{p_i{}^a}(-J(\varkappa_a{}^i, p_j{}^b)) \tag{2.10}$$

follows.

Its left-hand side is obviously the minimum value of the potential energy functional of a semilinear material. This follows from the fact that

$$\inf_{\varkappa_a{}^i}\inf_{x^j} I = \inf_{x^j}\inf_{\varkappa_a{}^i} I$$

Since

$$\inf_{\varkappa_a{}^i}\sup_{p_i{}^b}(-J) \geqslant \sup_{p_i{}^b}\inf_{\varkappa_a{}^i}(-J) = \sup_{p_i{}^a} J^\circ(p_i{}^a)$$

we obtain from (2.9)

$$\inf_{\varkappa_a{}^i}\sup_{p_j{}^b}(-J) \geqslant \sup_{p_i{}^b}\inf_{\varkappa_a{}^i}(-J) = \sup_{p_i{}^b} J^\circ(p_i{}^a)$$

The functional J° was constructed /4/ for the lower bound of the minimum value of the potential energy functional of a semilinear material. Since $\inf(-J) = -\sup(J)$, identity (2.8) can also be represented in the following form

$$\inf_{\varkappa_a{}^i}\inf_{x^j} I + \sup_{\varkappa_a{}^i}\inf_{p_j{}^b} J = 0$$

3. A certain property of the functionals I and J. Let $\bar{p}_i^a$ satisfy (1.1). We shall later call such an object a statically allowable Piola-Kirchhoff stress tensor. The anlogue of a kinematically allowable stress field for the functional $I(x^j)$ will be

$$q_i^a = \partial\Phi/\partial x_a^i\big|_{x'^i}$$

where x'^i is a certain kinematically allowable deformable position of the elastic body.

For any fixed $\varkappa_a^i$ the following identity holds:

$$I(\varkappa_a^i, x'^i) + J(\varkappa_a^i, \bar{p}_j^b) = J(\varkappa_a^i, \bar{p}_j^b) - J(\varkappa_a^i, q_i^b) - \int_{\partial V_0} P_i x'^i d\sigma + \int_{V_0} q_i^a x_a'^i d\tau \tag{3.1}$$

It follows from the fact that

$$I(\varkappa_a^i, x'^j) = \int_{V_0} \Phi\, d\tau - \int_{\partial V_0} P_i x'^i d\sigma = -\int_{V_0} \Phi^*\, d\tau + \int_{V_0} q_i^a x_a'^i d\tau - \int_{\partial V_0} P_i x'^i\, d\sigma$$

For any statically allowable field $\bar{p}_i^a$ the following relations is satisfied:

$$\int_{V_0} \bar{p}_i^a x_a'^i d\tau = \int_{\partial V_0} P_i x'^i\, d\sigma$$

This enables us to rewrite the identity (3.1) in the form

$$I(\varkappa_a^i, x'^j) + J(\varkappa_a^i, \bar{p}_b^j) = J(\varkappa_a^i, \bar{p}_b^j) - J(\varkappa_a^i, q_j^b) - \int_{V_0} (\bar{p}_i^q - q_i^a)\, x_a'^i d\tau \tag{3.2}$$

The latter is one of the possible forms of the Prager-Synge identity for the functional $I(x^j)$ /7/.

We will use the following notation: $\bar{U}^*(\sigma_{ab})$ is the Young-Fenchel transformation of the function $\bar{U}$ in (1.2) in $\rho_{ab} = \varkappa_a^i x_{ib} - g_{ab}^\circ$.

By virtue of (2.5) we have

$$\Phi^*(\varkappa_a^i, p_j^b) = V^*(p_i^a \varkappa_b^i) = V^*(\sigma_{ab}) = \bar{U}^*(\sigma_{ab}) + \sigma_a^a$$

Since $p_i^a x_a^i = \sigma^{ab}\rho_{ab} + \sigma_a^a$, it hence follows that

$$J(\varkappa_a^i, \bar{p}_j^b) - J(\varkappa_a^i, q_j^b) - \int_{V_0} (\bar{p}_i^a - q_i^a)\, x_a'^i\, d\tau = E_\varkappa^*(\bar{\sigma}_{ab}) - E_\varkappa^*(r_{ab}) - \tag{3.3}$$

$$\int_{V_0} (\bar{\sigma}_{ab} - r_{ab})\rho^{ab}\, d\tau = E_\varkappa^*(\bar{\sigma}_{ab} - r_{ab}), \quad E_\varkappa^*(\sigma_{ab}) = \int_{V_0} \bar{U}^*\, d\tau,$$

$$\bar{\sigma}^{ab} = \bar{p}_i^a \varkappa^{ib}, \quad r^{ab} = q_i^a \varkappa^{ib}$$

The relations

$$J(\varkappa_a^i, \bar{p}_j^b) - J(\varkappa_a^i, \overset{\circ}{p}{}_j^b) = E_\varkappa^*(\bar{\sigma}^{ab} - r^{\circ ab}) \tag{3.4}$$

$$I(\varkappa_a^i, x'^j) - I(\varkappa_a^i, \eta^{\circ j}) = E_\varkappa^*(r_{ab} - \overset{\circ}{r}_{ab}); \quad r^{\circ ab} = q_i^a \varkappa^{ib}\big|_{\eta^{\circ j}}$$

are established analogously.

Here $p_j^{\circ b}$ and $\eta^{\circ j}$ are minimizing elements of the functionals J and I. The identities (3.3) and (3.4) enable (3.2) to be represented in the form

$$E_\varkappa^*(\bar{\sigma}_{ab} - \overset{\circ}{r}_{ab}) + E_\varkappa^*(r_{ab} - \overset{\circ}{r}_{ab}) = E_\varkappa^*(\bar{\sigma}_{ab} - r_{ab}) \tag{3.5}$$

The functional $E_\varkappa^*$ is quadratic and positive-definite in the set of tensor functions of the second rank, and the relationship

$$c_1 \|\sigma^{ab}\|_{L_2(V_0)} \leqslant E_\varkappa^*(\sigma^{ab}) \leqslant c_2 \|\sigma^{ab}\|_{L_2(V_0)} = c_2 \int_V \sigma^{ab}\sigma_{ab} d\tau$$

holds for it, where $c_1 = 5/(24\mu)$ and $c_2 = 1/(4\mu)$, say. Since $\|\sigma_{ab}\|_{L_2(V_0)} = \|p_i^a\|_{L_2(V_0)}$, it follows from (3.4) that

$$\| \bar{p}_i^a - p_i^{\circ a} \|_{L_2(V_0)} \leqslant \frac{c_2}{c_1} \| \bar{p}_i^a - q_i^a \|_{L_2(V_0)}$$

$$\| q_i^a - p_i^{\circ a} \|_{L_2(V_0)} \leqslant \frac{c_2}{c_1} \| \bar{p}_i^a - q_i^a \|_{L_2(V_0)}$$

4. *The fundamental inequality.* Let $\bar{\varkappa}_a^i$ be an orthogonal tensor which makes the functional $J(\varkappa_a^i, \bar{p}_j^b)$ reach a maximum for fixed $\bar{p}_j^b$. This means that for any $\varkappa_a^i$

$$J(\varkappa_a^i, \bar{p}_j^b) \leqslant J(\bar{\varkappa}_a^i, \bar{p}_j^b) \tag{4.1}$$

Furthermore, we assume the tensor $\bar{p}_i^a$ to be statically allowable. If x_0^i is a minimizing element of the functional $I^\circ(x^j)$ and $\varkappa_{0a}^i$ is the orthogonal part of its distortion, then

$$J(\varkappa_{0a}^i, p_{0j}^b) + I(\varkappa_{0a}^i, x_0^j) = 0$$

Here p_{0j}^b is the Piola-Kirchhoff stress tensor corresponding to the deformed position of the elastic body x_0^i. Taking the above into account, the identity (3.1) can be rewritten in the form

$$I(\bar{\varkappa}_a^i, x'^j) - J(\varkappa_{0a}^i, x_0^j) + J(\bar{\varkappa}_a^i, \bar{p}_j^b) - J(\varkappa_{0a}^i, p_{0b}^j) = \tag{4.2}$$
$$J(\bar{\varkappa}_a^i \bar{p}_j^b) - J(\bar{\varkappa}_a^i, q_j^b) - \int_{V_0} (\bar{p}_j^b - q_j^b) x_b'^j d\tau$$

Since $I(\varkappa_{0a}^i, x_0^j) \leqslant I(\bar{\varkappa}_a^i, x'^j)$, then by virtue of (3.3) and (3.4) we obtain from (4.2)

$$E^*_{\varkappa_0}(\bar{\sigma}^{ab} - \sigma^{\circ ab}) \leqslant E^*_{\bar{\varkappa}}(\bar{\sigma}^{ab} - r^{ab}) \tag{4.3}$$

Finally, on the basis of (3.5) we have the inequality

$$\| \bar{p}_i^a - p_{0i}^p \|_{L_2(V_0)} \leqslant \frac{c_2}{c_1} \| \bar{p}_i^q - q_i^q \|_{L_2(V_0)} \tag{4.4}$$

which is the main purpose of this paper. As in geometrically linear problems it enables us to estimate the difference between the statically allowable Piola-Kirchhoff stress field $\bar{p}_i^a$ and the minimizing element of the potential energy functional of a semilinear elastic body p_{0i}^a in the norm L_2 in terms of the difference $\bar{p}_i^a - q_i^a$. The stress field q_i^a here is not kinematically allowable as it would be in the geometrically linear problem. Its difference from the true kinematically allowable field corresponding to the deformed state x'^i is that the latter is calculated from (3.1) with $\Phi = \Phi(\varkappa_a'^i, x^j)$, where $\varkappa_a'^i$ is the orthogonal part of the distortion of $x_a'^i$ and the second with $\Phi(\bar{\varkappa}_a^i, x^j)$.

For applications of the inequality (4.4), it is merely necessary to indicate the form $\bar{\varkappa}_a^i$ explicitly, which depends on $\bar{p}_i^a$.

The upper bound of the functional $J(\varkappa_a^i, \bar{p}_j^b)$ in all $\varkappa_a^i$ reduces in the case under consideration to point-by-point maximization of the function $\Phi^*(\varkappa_a^i, \bar{p}_j^b)$. This has the form (2.6). Since $\bar{\sigma}^{ab}\bar{\sigma}_{ab} = \bar{p}_i^a p_a^i$ is independent of $\varkappa_a^i$, then finding $\bar{\varkappa}_a^i$ will reduce to finding the maximizing element of the function

$$\psi = -\frac{1}{4\mu} \frac{\lambda(\lambda + 2\mu)}{(3\lambda + 2\mu)^2} (\bar{\sigma}_a^a)^2 + \bar{\sigma}_a^a$$

where, as before, $\bar{\sigma}_a^a = \bar{p}_i^a \varkappa_a^i$. It is clear that $|\bar{\sigma}_a^a| \leqslant |\bar{p}|_a^a$.

The graph of the dependence of ψ on $z = \sigma_a^a$ is a quadratic parabola with apex at the point $z_0 = 2E\nu^{-1}(1 + 2\lambda/(\lambda + 2\mu))$. If the stresses under consideration in the semilinear material are such that $|p|_a^a \leqslant z_0$, then ψ takes it maximum value at the maximum of $\bar{\sigma}_a^a$. The answer to the question for which $\varkappa_a^i$ the quantity $\bar{\sigma}_a^a$ is a maximum is given by Lemma 2. We obtain that if $\det \| \bar{p}_i^a \| \geqslant 0$ then $\bar{\varkappa}_i^a = \mu_i^a$ and μ_i^a is the orthogonal part of the tensor $\bar{p}_i^a$. In the case when $\det \| p_i^a \| < 0$, $\bar{\varkappa}_i^a = \mu_i^b s_b^{ca}$, where $s_b^{\circ a}$ is an orthogonal matrix selected from the condition $\max_{s_{ab}} \{ |\bar{p}|_b^a s_a^b \}$, the maximum is sought in orthogonal matrices that have the form (1.6) in the principal coordinate system for the tensor $|\bar{p}|_b^a$.

Finally we obtain that the following theorem holds.

Theorem 2. Let $\bar{p}_i^a$ be a certain Piola-Kirchhoff stress field satisfying the first two equations of (1.1) and let q_i^a be the stress field associated with an arbitrary deformed state of the elastic body by the relationship

$$q_i^a = \partial\Phi(\bar{\varkappa}_a^i, x^i)/\partial x_a^i |_{x^i = x'^i}$$

Then the minimizing element x_0^i of the potential energy functional $I^\circ(x^i)$ of a semilinear material is connected with the tensors $\bar{p}_i^a$ and q_i^a the relationship (4.4), where

$$p_{0i}^a = \partial U/\partial x_a^i|_{x^i=x_0^i}$$

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ON THE REALIZATION OF HOLONOMIC CONSTRAINTS*

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The idea of realizing holonomic constraints by means of elastic forces was proposed by Lecornu, Klein and Prandtl /1/ when dealing with the paradox of dry friction discovered by Painleve. The general theorem on the realization of holonomic constraints with the help of elastic forces directed towards the configurational manifold of a constrained system was proposed by Courant and was proved in /2/. The generalization of Courant's theorem was considered in /3-5/ by studying the passage to the limit in the case when the velocity of the system at the initial instant is transverse to the manifold defined by the constraint equations. In /2-5/ the assumption that the system in question in conservative is used to a considerable degree.

The main results of the present paper is the fact that the theorem on the passage to the limit holds without assuming that the generalized forces are potential in character. The elastic forces acting on the "free" system have, in general, no limit when the coefficient of elasticity tends to infinity. However, as is shown below, after suitable regularization these forces tend precisely to the reactions of the system with constraints.

1. Initial equations. Let a natural mechanical system be given in $\mathbf{R}^n = \{r\}$, constrained by n_1 ideal holonomic constraints. Let $E(r^\cdot, r)$ be the kinetic energy of the system without constraints and let $F(r^\cdot, r)$ be the generalized active force. The equations of motion will have the form

$$(\partial E/\partial r^\cdot)^\cdot - \partial E/\partial r = F + R \tag{1.1}$$

where R is the reaction force of the constraints. The constraints define in $\mathbf{R}^n$ a manifold M of dimensions $n_0 = n - n_1$, over which the system must move. In accordance with the axiom of the ideality of the constraints, the 1-form Rdr vanishes on the vectors tangent to M.

We shall consider the problem of realizing the constraints using the force with potential

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